

JOURNAL OF FUNCTIONAL ANALYSIS 51, 229–258 (1983)

One-Dimensional Schrödinger Operators with Random or Deterministic Potentials: New Spectral Types

RENÉ CARMONA*

*Department of Mathematics, University of California at Irvine, Irvine, California 92717**Communicated by the Editors*

Received December 13, 1982

We prove sufficient conditions involving only potential asymptotic near one of the infinities in order to have purely absolutely continuous components in the spectrum. These deterministic results are then applied to random cases and exhibit classes of models for which, with probability one, one component of the spectrum is purely absolutely continuous and the rest is dense pure point with exponentially decaying eigenfunctions.

I. INTRODUCTION

This paper should be regarded as a new contribution to the study of the spectral properties of random self-adjoint operators on $L^2(R, dt)$ of the form

$$H(\omega) = -\frac{d^2}{dt^2} + q(t, \omega),$$

where $\{q(t, \omega); t \in R, \omega \in \Omega\}$ is a stochastic process on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ whose properties will be specified later on. These Schrödinger operators play a crucial role in the mathematical theory of quantum mechanical disordered systems. They exhibit spectral characteristics that are drastically different from those of the corresponding deterministic operators. For example, it has been shown that for some Brownian motion models, \mathbb{P} -almost surely the spectrum of H is pure point ([11]) with exponentially decaying eigenfunctions ([14]), the exact rate of exponential fall off being given by the upper Ljapunov index of the corresponding Cauchy problem ([2, 6]). Such properties are highly unexpected under the assumptions that were used so far in the deterministic cases!

* Partially supported by NSF Grant M.C.S. 82-02045.

In higher dimensions it can be inferred from the works of physicists that the spectrum will not always be pure point, but will have subsets on which it is purely absolutely continuous (corresponding to the allowed energies of diffusion) and subsets on which it is pure point with exponentially decaying eigenfunctions (corresponding to the allowed energies of localization). Our objective is to exhibit wide classes of random potentials $q(t, \omega)$ for which this is already the case in one dimension. These potentials are not stationary (as stochastic processes) on the whole real line but merely on a half axis. They have different asymptotic behaviors when $t \rightarrow -\infty$ and $t \rightarrow +\infty$. In fact to prove the existence almost sure of a fixed part of the spectrum where the latter is purely absolutely continuous we work with the random element $\omega \in \Omega$ fixed. This means that we prove results on the absolute continuity of the spectrum for deterministic Schrödinger operators. Even though this sort of problem has been extensively studied for a long time, our results are surprisingly new. Their novelty and their power rely on the fact that they require only the knowledge of the asymptotic behavior of the potential $q(t)$ when $t \rightarrow -\infty$ or $t \rightarrow +\infty$, and not of both as usual. See Theorems II.4 and II.9 which we regard as the main results of this paper. This achievement is made possible by the following trick: The information on the spectral characteristics of the operators is extracted from the Lebesgue decomposition of the spectral measure given by the classical theory of self-adjoint differential equations see, for example, [4, 13]. This measure is constructed as the vague limit of the spectral measures of the eigenvalue problems in finite intervals with appropriate boundary conditions. The latter disappear in the limiting process, so we can regard them as irrelevant in the limit. Hence we can safely integrate out one of them. This simple operation (which should be innocent—at least in the limit) turns the *pure point* spectral measures of the eigenvalue problem in bounded intervals into *absolutely continuous measures*. Moreover, and maybe most of all, their densities have simple forms in terms of the norms of the solutions of the corresponding Cauchy problems and they separate the contributions of the values of $q(t)$ for $t \geq 0$ from those of $q(t)$ for $t \leq 0$.

All our results in the deterministic case follow, in a very simple manner, this “magic” formula. They are contained in Section II and they can be read without prior knowledge or interest in stochastic processes since they are purely functional analytic.

At this point we would like to make two remarks. First, we note that the absolute continuity of the spectrum of one-dimensional Schrödinger operators has already been investigated when the potential $q(t)$ has different asymptotics when $t \rightarrow -\infty$ and $t \rightarrow \infty$ (see [7]). This study was conducted in the more general framework of time evolution in scattering theory. Once reinterpreted in the present context, these results appear to be weaker than ours because they require the knowledge of $q(t)$ both when $t \rightarrow +\infty$ and

$t \rightarrow -\infty$ and because they show only the existence of an absolutely continuous component in the spectrum without proving, like we do, there is no other one. Second, we notice that changes of variables using randomness of the boundary conditions have already been used to study random Schrödinger operators or their difference analogs (see, for example, [12, 15]) but it seems that it is the first time that this idea is pushed far enough to lead to an explicit formula for the density of the integrated spectral measure

As an illustration of the kind of problems that can be handled by our method let us assume that $q(t)$ is (a) integrable near $-\infty$, or (b) periodic on $(-\infty, 0]$ and *ANYTHING* on \mathbb{R}_+ provided that the operator $H = -(d^2/dt^2) + q(t)$ is in the limit point case at $+\infty$. We prove that the spectrum of H contains $(0, \infty)$ and is purely absolutely continuous there in case (a) and that, in case (b) the spectrum that H would have if $q(t)$ was extended periodically to the whole line is contained in the actual spectrum of H and the latter is purely absolutely continuous there. Moreover, it was pointed out to us by Barry Simon that our approach could be used to streamline part of the work of Dinaburg and Sinai (namely, [8, Theorem 2]) on the existence of an absolutely continuous component in the spectrum of some one-dimensional Schrödinger operators with quasi-periodic potentials.

These results are proved by checking that the first order vector differential equation associated to the second order Schrödinger eigenvalue equation is stable uniformly when the eigenvalue is restricted to compact sets (see Theorem II.4 and its corollaries).

In fact these properties are reminiscent of the following intuitive idea: Our hypotheses essentially ensure that on one half line the solutions of the Schrödinger equation in a given energy set are plane waves, and then somehow tunneling effects should force the diffusion to infinity and the absolute continuity of the spectrum. Moreover, when the potential $q(t)$ is smooth enough, it is very easy to prove that there are no square integrable eigenfunctions, so that the real mathematical essence of what we are doing is proving that there is no singular continuous spectrum. Also we note that local uniform stability is far from being necessary. Indeed we show in Theorem II.9 and its corollary that our approach can be used to handle cases where this assumption is not satisfied. The extra ingredient in the proof is the following: We do not restrict ourselves to making random one of the two boundary conditions, we also randomize the size of the bounded intervals that eventually cover the whole real line \mathbb{R} in the limiting process involved in the construction of the spectral measure. As a consequence we can reprove and improve the deterministic result of [1] where it was shown that the spectrum of one-dimensional Schrödinger operators with an electric field and bounded random or deterministic potentials is purely absolutely continuous. Let us emphasize that our new result is independent of the behavior of the potential and the field on a half axis.

The proof of this last application (namely, Corollary II.10) is rather lengthy. Since it is technical and presumably not very instructive, we urge the reader to skip it for a first reading and to concentrate on the proof of Theorems II.4 and II.9 in order to appreciate the simplicity of the idea.

The last part of the paper, Section III, deals with the study of random cases. It heavily relies on both the results of Section II, and on the approach to the random case presented in [2]. Indeed they both integrate separately the properties of the potential $q(t)$ for $t \leq 0$ and $t \geq 0$. The first ones (for example) will provide the pure absolute continuity of the spectrum in certain (energy bands) subsets of the spectrum via the results of Section II, while the second ones will force the pure point character on the rest of spectrum via the techniques of [2].

As an example let $\varepsilon > 0$ and let us denote by $\{\varepsilon(t, \omega); t \geq 0, \omega \in \Omega\}$ a stationary process of Brownian motion on $[-\varepsilon, +\varepsilon]$ with periodic boundary condition. Then if $\{q(t, \omega); t \in \mathbb{R}, \omega \in \Omega\}$ is any stochastic process which coincides with $\varepsilon(t, \omega)$ for $t \geq 0$ and which has almost surely integrable paths near $-\infty$ and if these paths are almost surely bounded below by $-\varepsilon$ on $(-\infty, 0)$, then, for \mathbb{P} -almost all $\omega \in \Omega$ the spectrum of $H(\omega)$ is $[-\varepsilon, +\infty)$ as a set, it is purely absolutely continuous on $(0, \infty)$ and there exists a countable set of eigenvalues, dense in $[-\varepsilon, 0]$, for which the corresponding eigenfunctions decay exponentially at $+\infty$ and $-\infty$ according to the Ljapunov indexes of the corresponding Cauchy problems.

This corresponds to case (a) of our above discussion of deterministic problems. Regarding case (b) we have: If $q(t, \omega) = q_1(t) + \tilde{\varepsilon}(t, \omega)$, where $q_1(t)$ is a periodic function independent of $\omega \in \Omega$ and where $\tilde{\varepsilon}(t, \omega) = \varepsilon(t, \omega)$ if $t \geq 0$ and 0, otherwise, then for \mathbb{P} -almost all $\omega \in \Omega$ the spectrum of $H(\omega)$ is equal to $\Sigma = \bigcup_i [a_i - \varepsilon, b_i + \varepsilon]$, where $\Sigma_1 = \bigcup_i [a_i, b_i]$ is the spectrum of the (nonrandom) operator $H_1 = -(d^2/dt^2) + q_1(t)$, it is purely absolutely continuous on Σ_1 and dense pure point as above on $\Sigma \setminus \Sigma_1$. This example has the simple interpretation of a mathematical model for a quantum, one-dimensional perfect crystal with impurities only on the right-hand half.

The study of these random models is completely new and it may look very intriguing at first sight. Unfortunately, the type of randomness (i.e., minimal sets of assumptions on the stochastic process $\{q(t, \omega); t \in \mathbb{R}, \omega \in \Omega\}$) leading to these results is not well understood yet. Of course, it is easy to go beyond the Brownian motion examples given above. Nevertheless, the proofs and the statements of theorems have to be notably lengthened and complicated to gain only a slight generalization in the type of randomness. The state of the art is not satisfactory in this domain. So we refrain from aiming at a great generality because we do not want to pay the price for it. Our potentials will be the natural abstractions of the above Brownian motion. They have been introduced by the Russian school in [11, 14] and used since then. (See, nevertheless, [1, Section III] and the last part of its

introduction for other classes of random potentials which can be handled in the same way, and for further comments on the various sets of assumptions). Hence we accept to take the chance to have our results regarded as mathematical curiosities rather than general results, as we believe they are.

Finally we close this introduction with the following claim: Even though we will only deal in this paper with Schrödinger operators on $L^2(\mathbb{R})$ as second order differential operators on the real line, all the results we prove, except Corollary II.8, obtain in the case of the corresponding finite difference operators on $l^2(\mathbb{Z})$. The modifications are minor and the proofs are less technical may be, but we will not pursue their discussions.

II. DETERMINISTIC CASE

The main results of this section are Theorems II.4 and II.9. Their proofs require some preparatory work and notations. Lemma II.1 is essentially known and should be considered as belonging to the folklore. We include it with a complete proof because we could not find it in the literature in a convenient form. The proof is very simple and its idea seems to go back to an old paper of Sch'nol (see [16; 17, Sect. C4]).

LEMMA II.1. *Let $\lambda \in \mathbb{R}$ and let $q(t)$ be a real-valued locally integrable function which is bounded from below near $-\infty$ and $+\infty$ by a function of the form $-a_0(t^2 + 1)$ for some constant $a_0 > 0$ and such that all the solutions of $-y''(t) + (q(t) - \lambda)y(t) = 0$ and their derivatives are bounded in a neighborhood of $-\infty$ or $+\infty$. Then λ belongs to the spectrum of the unique self-adjoint extension of the operator defined on $C_0^\infty(\mathbb{R})$ by $[Hf](t) = -f''(t) + q(t)f(t)$.*

Proof. It is well known that the operator H defined in the statement of the lemma is essentially self-adjoint because it is in the so-called limit point case (see [4, 13], for example). Moreover our assumptions and [13, Theorem 2.3, p. 111] imply the existence of a solution, say φ , of $-y''(t) + (q(t) - \lambda)y(t) = 0$ which, together with its derivative is bounded near $+\infty$ (the proof is similar if our assumption works for $-\infty$) and which is not square integrable at $+\infty$. Now we set $\varphi_r = \varphi j_r / \|\varphi j_r\|$, where $j_r \in C_0^\infty(\mathbb{R})$ is supported in $[0, r + 1]$, is identically equal to 1 on $[1, r]$ and is such that $\sup_{t \in \mathbb{R}, r \geq 1} \max(|j_r(t)|, |j_r'(t)|, |j_r''(t)|) < +\infty$. Then

$$-\varphi_r''(t) + (q(t) - \lambda)\varphi_r(t) = \frac{1}{\|\varphi j_r\|} [-j_r''(t)\varphi(t) - 2j_r'(t)\varphi'(t)]$$

so that the L^2 -norm of $-\varphi_r'' + (q - \lambda)\varphi_r$ is bounded above by a constant divided by the L^2 -norm of φj_r . The latter goes to infinity when r tends to

infinity by construction of φ and the j_r 's. Hence we can conclude that λ belongs to the spectrum of H . ■

Lemma II.2 is, by far, more involved than Lemma II.1. It should be regarded as the main technical device of the paper. Moreover, it requires some notation which will be used throughout the rest of the paper. So first we proceed to introducing them. In order to study the spectral characteristics of the operator

$$H = -\frac{d^2}{dt^2} + q(t) \quad (\text{II.1})$$

on the whole real line \mathbb{R} , where $q(t)$ is a locally integrable and locally bounded real valued function, we first study it on bounded intervals $[a, b]$ such that $a < 0 < b$. Equation (II.1) has to be supplemented by boundary conditions at a and b in order to define a self-adjoint operator on $L^2([a, b])$. We will choose separated boundary conditions: they are determined by two angles α and β which are only defined modulo π and they have the form

$$y(a) \cos \alpha - y'(a) \sin \alpha = 0, \quad y(b) \cos \beta - y'(b) \sin \beta = 0. \quad (\text{II.2})$$

Taking into account these boundary conditions the eigenvalue equation

$$-y''(t) + (q(t) - \lambda)y(t) = 0 \quad (\text{II.3})$$

becomes a regular self-adjoint boundary problem and a spectral measure $\sigma_{a,b,\alpha,\beta}$ is associated to it in the following way:

$$\sigma_{a,b,\alpha,\beta} = \Sigma \left(\int_a^b |y_\lambda(t)|^2 dt \right)^{-1} \delta_\lambda, \quad (\text{II.4})$$

where δ_λ denotes the unit mass at the point $\lambda \in \mathbb{R}$, where the summation is over the eigenvalues—namely, those $\lambda \in \mathbb{R}$ for which there exists a solution of (II.3), which satisfies the boundary conditions (II.2)—and where $y_\lambda(t)$ is any of these solutions which has amplitude 1 for $t = 0$. The value at $t \in \mathbb{R}$ of the amplitude of a solution $y(t)$ of (II.3) is defined by

$$r(t) = [|y(t)|^2 + |y'(t)|^2]^{1/2}. \quad (\text{II.5})$$

If $q(t)$ satisfies some very mild conditions for t near infinities, it is well known that the measure $\sigma_{a,b,\alpha,\beta}$ converges vaguely, as $a \rightarrow -\infty$ and $b \rightarrow +\infty$, toward a measure σ independent of the choice of the boundary conditions α and β , and this measure contains all the spectral informations we want to

know on H (see, for example, [3, 4, 13]). In order to exploit this independence we will try to substitute the measure

$$\sigma_{a,b,\beta} = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \sigma_{a,b,\alpha,\beta} d\alpha \quad (\text{II.6})$$

whenever possible. Indeed Lemma II.2 shows that this measure is absolutely continuous with respect to Lebesgue's measure, and we can identify the Radon–Nikodym derivative in terms of the amplitude of some particular solutions of the second order differential equation (II.3). Consequently, this density can be expected to be controlled by results from the stability theory of ordinary differential equations, yielding estimates on σ after taking the limits $a \rightarrow -\infty$ (see [3, 5, 10]). This is the core of our approach. The special role played by 0 in the construction of “the” spectral measure σ is sometimes emphasized by saying that σ is constructed with reading point 0.

LEMMA II.2.

$$\sigma_{a,b,\beta}(d\lambda) = \frac{1}{\pi} e^{\int_a^0 [1+q(u)-\lambda] \sin 2\theta_{\lambda,0}(u, \theta_{\lambda,b}(0,\beta)) du} d\lambda, \quad (\text{II.7})$$

where $\theta_{\lambda,x}(t,\varphi)$ denotes the value at t of the solution of the differential equation

$$\theta'(t) = 1 + [\lambda - 1 - q(t)] \sin^2 \theta(t) \quad (\text{II.8})$$

satisfying the initial condition $\theta(x) = \varphi$.

The following comments seem necessary in order to illuminate the meaning of the fundamental formula (II.7).

Up to the factor $1/\pi$, the right-hand side is the inverse of the square of the amplitude computed at a , of the solution of $-y'' + (q - \lambda)y = 0$ which has amplitude 1 and phase $\theta_{\lambda,b}(0,\beta)$ at the origin. What is remarkable is the splitting of the influences of the potential $q(t)$ for $t \geq 0$ and $t \leq 0$. The values of $q(t)$ for $t \geq 0$, the value of b and the boundary condition β enter only in the computation of $\theta_{\lambda,b}(0,\beta) = \theta$, and once this phase θ is known the density of $\sigma_{a,b,\beta}$ depends exclusively on the values of $q(t)$ for $t \leq 0$. So, if we can control the amplitude $r_\lambda(t)$ for $t \leq 0$ independently of the phase at the origin, we will be able to control the “spectral measure $\sigma_{a,b,\beta}$ ” and its limit σ without requiring any assumption on the values of the potential $q(t)$ for $t \geq 0$.

This remark paves the way to our main deterministic result (see Theorem II.4) and its various applications, deterministic or not.

Proof. In the following proof the numbers a , b , and β will remain fixed.

Moreover, we may without any loss of generality prove (II.7) only for λ varying in a bounded interval \mathcal{A} . Using the well-known Prüfer transformation

$$y(t) = r(t) \sin \theta(t), \quad y'(t) = r(t) \cos \theta(t) \quad (\text{II.9})$$

in (II.3), we see that the so-called phase $\theta(t)$ has to satisfy (II.8) and consequently $\lambda \in \mathbb{R}$ is an eigenvalue of problem (II.3)–(II.2), and so contributes to the summation in (II.4), if and only if

$$\theta_{\lambda,a}(b, \alpha) \in \{\beta + k\pi; k \in \mathbb{Z}\}. \quad (\text{II.10})$$

Because of the unicity of solutions of (II.8) we have

$$\theta_{\lambda,a}(t, \alpha) + k\pi = \theta_{\lambda,a}(t, \alpha + k\pi)$$

for all $t \in \mathbb{R}$ and for all $k \in \mathbb{Z}$. Moreover it is easy to check that

$$\frac{\partial \theta_{\lambda,a}(t, \alpha)}{\partial \alpha} = e^{\int_a^t [\lambda - 1 - q(u)] \sin 2\theta_{\lambda,a}(u, \alpha) du} \quad (\text{II.11})$$

which implies, first that $\partial \theta_{\lambda,a}(t, \alpha) / \partial \alpha > 0$, and second that

$$\theta_{\lambda,a}(t, (-\pi/2, \pi/2]) = (\theta_{\lambda,a}(t, -\pi/2), \theta_{\lambda,a}(t, -\pi/2) + \pi]. \quad (\text{II.12})$$

This implies that for each λ we have one (and exactly one) $\alpha \in (-\pi/2, \pi/2]$ for which (II.10) holds. Now an elementary computation shows that

$$\frac{\partial \theta_{\lambda,a}(b, \alpha)}{\partial \lambda} = \int_a^b \sin^2 \theta_{\lambda,a}(s, \alpha) e^{\int_s^b [\lambda - 1 - q(u)] \sin 2\theta_{\lambda,a}(u, \alpha) du} ds \quad (\text{II.13})$$

which shows that this derivative is bounded by a constant, say k , uniformly in α and λ since the latter varies in compact sets and since the function $q(t)$ is bounded on $[a, b]$ which is held fixed. We will use this fact in the following form:

“given α and an interval $\mathcal{A}' \subset \mathcal{A}$ of length less than π/k , there exists at most one $\lambda \in \mathcal{A}'$ such that (II.10) holds.”

Let us assume that such an open interval \mathcal{A}' is fixed for a while. Now let $\alpha \in (-\pi/2, \pi/2]$ be such that there exist $\lambda(\alpha) \in \mathcal{A}'$ and $k(\alpha) \in \mathbb{Z}$ such that

$$\theta_{\lambda(\alpha),a}(b, \alpha) = \beta + k(\alpha) \pi.$$

If we define the function F_α by

$$F_\alpha(\lambda, \alpha') = \theta_{\lambda, \alpha}(b, \alpha') - \beta - k(\alpha) \pi$$

we have $\partial F_\alpha(\lambda(\alpha), \alpha)/\partial \lambda > 0$ by (II.13) and we can solve the equation $F_\alpha(\lambda, \alpha') = 0$ by the implicit function theorem. Hence, there exist an open neighborhood $I(\alpha)$ of α and a continuously differentiable function $I(\alpha) \ni \alpha' \mapsto \lambda(\alpha') \in A'$ such that

$$\forall \alpha' \in I(\alpha), \quad F_\alpha(\lambda(\alpha'), \alpha') = 0.$$

This proves in particular that the set of such α 's is open and so, it can be written as the at most countable union of disjoint open intervals I_i 's. Now, for each i and for each $\alpha \in I_i$ we know that there exists exactly one $\lambda_i(\alpha)$ and one $k_i(\alpha)$ such that $\theta_{\lambda_i(\alpha), \alpha}(b, \alpha) = \beta + k_i(\alpha) \pi$. Consequently, we have a well-defined function $\bigcup_i I_i \ni \alpha \rightarrow \lambda(\alpha)$ given by $\lambda(\alpha) = \lambda_i(\alpha)$ if $\alpha \in I_i$. This map is one-to-one and continuously differentiable. Moreover, the $\lambda(I_i)$ are disjoint and cover A' .

Next let f be any continuous function on \mathbb{R} , the support of which is contained in A' . We have

$$\begin{aligned} \int_A f(\lambda) \sigma_{a, b, \alpha, \beta}(d\lambda) &= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} d\alpha \int_{A'} f(\lambda) \sigma_{a, b, \alpha, \beta}(d\lambda) \\ &= \frac{1}{\pi} \sum_i \int_{I_i} d\alpha \int_{A'} f(\lambda) \sigma_{a, b, \alpha, \beta}(d\lambda). \end{aligned} \quad (\text{II.14})$$

For each i we perform the change of variables $\lambda = \lambda_i(\alpha)$ in the integral $\int_{I_i} d\alpha$ because we know that, if $\alpha \in I_i$ there exists one and exactly one $\lambda \in A'$, namely, $\lambda_i(\alpha)$ which is charged by the spectral measure $\sigma_{a, b, \alpha, \beta}$, so the integral with respect to the latter reduces to $f(\lambda_i(\alpha))$ times the weight given by the point measure $\sigma_{a, b, \alpha, \beta}$ to the eigenvalue $\lambda_i(\alpha)$. We define the function $\sigma_{a, b, \alpha, \beta}$ by

$$\sigma_{a, b, \alpha, \beta}(\lambda) = \left[\int_a^b r_\lambda(s)^2 \sin^2 \theta_\lambda(s) ds \right]^{-1}, \quad (\text{II.15})$$

where $r_\lambda(s)$ and $\theta_\lambda(s)$ denote the amplitude and the phase of any solution of $-y'' + (q - \lambda)y = 0$ whose phase at a is α and amplitude 1 at the origin. This function is defined for all λ and coincide with the weight of the spectral measure $\sigma_{a, b, \alpha, \beta}$ when λ is an eigenvalue of problem (II.3)–(II.2). By the definition of the Prüfer transformation (II.9), r_λ must be a solution of the first order differential equation

$$r'(t) = \frac{1}{2} r(t) [1 + q(t) - \lambda] \sin 2\theta_\lambda(t) \quad (\text{II.16})$$

which can be integrated provided we assume the knowledge of $\theta_\lambda(t)$. This makes possible the rewriting of (II.15) in the form

$$\sigma_{a,b,\alpha,\beta}(\lambda) = \left[\frac{\int_a^b \sin^2 \theta_{\lambda,a}(s, \alpha) e^{f_a^s[1+q(u)-\lambda] \sin 2\theta_{\lambda,a}(u, \alpha)} du}{e^{f_a^b[1+q(u)-\lambda] \sin 2\theta_{\lambda,a}(u, \alpha)}} ds \right]^{-1}. \quad (\text{II.17})$$

Now, the implicit function theorem gives

$$\begin{aligned} \left| \frac{1}{\lambda'_i(\alpha)} \right| &= \left| \frac{\partial F_i}{\partial \lambda}(\lambda_i(\alpha), \alpha) \frac{\partial F_i}{\partial \alpha}(\lambda_i(\alpha), \alpha) \right| \\ &= \left| \frac{\partial \theta_{\lambda,a}(b, \alpha)}{\partial \lambda} \left(\frac{\partial \theta_{\lambda,a}(b, \alpha)}{\partial \alpha} \right)^{-1} \right|_{\lambda=\lambda_i(\alpha)} \\ &= \left| \frac{\int_a^b \sin^2 \theta_{\lambda,a}(s, \alpha) e^{f_a^b[\lambda-1-q(u)] \sin 2\theta_{\lambda,a}(u, \alpha)} du}{e^{f_a^b[\lambda-1-q(u)] \sin 2\theta_{\lambda,a}(u, \alpha)}} ds \right|_{\lambda=\lambda_i(\alpha)} \\ &= \int_a^b \sin^2 \theta_{\lambda,a}(s, \alpha) e^{f_a^s[1+q(u)-\lambda] \sin 2\theta_{\lambda,a}(u, \alpha)} du ds \Big|_{\lambda=\lambda_i(\alpha)}, \quad (\text{II.18}) \end{aligned}$$

where we used (II.11) and (II.13). Putting together (II.17) and (II.18) we obtain

$$\begin{aligned} \frac{1}{|\lambda'_i(\alpha)|} \sigma_{a,b,\alpha,\beta}(\lambda_i(\alpha)) \\ = \exp \left[\int_a^0 [1+q(u)-\lambda] \sin 2\theta_{\lambda,a}(u, \alpha) du \right] \Big|_{\lambda=\lambda_i(\alpha)}. \quad (\text{II.19}) \end{aligned}$$

Note that in order to perform the change of variables $\lambda = \lambda_i(\alpha)$, we have to replace α in formula (II.19) by $\alpha_i(\lambda)$, where α_i is the inverse function of $I_i \ni \alpha \rightarrow \lambda_i(\alpha) \in \lambda_i(I_i)$. At this point it is more advantageous to remark that since $\lambda = \lambda_i(\alpha)$ is an eigenvalue for problem (II.3)–(II.2), the two boundary conditions are satisfied and we have

$$\theta_{\lambda_i(\alpha),a}(u, \alpha) = \theta_{\lambda_i(\alpha),0}(u, \theta_{\lambda_i(\alpha),b}(0, \beta))$$

by unicity of solutions of Cauchy problems for ordinary differential equations. Taking this last point into account we get

$$\begin{aligned} \int_{I_i} d\alpha \int_{\Lambda} f(\lambda) \sigma_{a,b,\alpha,\beta}(d\lambda) \\ = \int_{\lambda_i(I_i)} f(\lambda) \exp \left[\int_a^0 [1+q(u)-\lambda] \sin 2\theta_{\lambda,a}(u, \theta_{\lambda,b}(0, \beta)) du \right] d\lambda \end{aligned}$$

which, once plugged into (II.14) gives the desired result since the $\lambda_i(I_i)$ are disjoint and cover \mathcal{A}' . For the same formula of change of variables to obtain when the diameter of the support of f is no longer assumed to be less than π/k an argument using partition of unity can be used. The proof is now complete. ■

The $\lambda \in \mathbb{R}$ such that all the solutions of

$$-y'' + (q - \lambda)y = 0 \quad (\text{II.20})$$

and their derivatives are bounded in a neighborhood of $+\infty$ (resp. $-\infty$) are those $\lambda \in \mathbb{R}$ for which

$$\sup_{t > 0} \|U_\lambda(t, 0)\| < \infty \quad (\text{resp. } \sup_{t < 0} \|U_\lambda(t, 0)\| < \infty), \quad (\text{II.21})$$

where $\{U_\lambda(t, s); t, s \in \mathbb{R}\}$ denotes the propagator of the vector differential equation

$$Y'(t) = \begin{bmatrix} 0 & 1 \\ q(t) - \lambda & 0 \end{bmatrix} Y(t). \quad (\text{II.22})$$

They are called stability points for (II.20) at $+\infty$ (resp. $-\infty$). The standard terminology is that the second order differential equation (II.20) or the vector differential equation (II.22) is stable at $+\infty$ (resp. $-\infty$). These λ 's were shown to belong to the spectrum of the corresponding Schrödinger operator. In order to exhibit subsets of this spectrum where the spectral measure is purely absolutely continuous we need a more restrictive notion of stability.

DEFINITION II.3. $\mathcal{A} \subset \mathbb{R}$ is said to be a set of local uniform stability near $+\infty$ (resp. $-\infty$) for (II.20) or equivalently for (II.22), if for each $\lambda \in \mathcal{A}$ there exists an open neighborhood V of λ such that

$$\sup_{t > 0, \lambda' \in \mathcal{A} \cap V} \|U_{\lambda'}(t, 0)\| < +\infty \quad (\text{resp. } \sup_{t < 0, \lambda' \in \mathcal{A} \cap V} \|U_{\lambda'}(t, 0)\| < +\infty). \quad (\text{II.23})$$

Note that if \mathcal{A} is an open set of local uniform stability near $+\infty$, then, for any compact set K contained in \mathcal{A} , we can find a finite constant $k = k(K)$ such that

$$\sup_{t > 0, \lambda \in K} \|U_\lambda(t, 0)\| \leq k.$$

Obviously the same result holds for the local uniform stability near $-\infty$.

THEOREM II.4. *Let $q(t)$ be a real-valued locally integrable function on \mathbb{R} which is bounded below near $+\infty$ and $-\infty$ by a function of the form $-a_0(t^2 + 1)$ for some constant $a_0 > 0$. Then any open set Λ_0 of local uniform stability for (II.20) is contained in the spectrum of the unique self-adjoint extension in $L^2(\mathbb{R})$ of the operator H defined for $f \in C_0^\infty(\mathbb{R})$ by $[Hf](t) = -f''(t) + q(t)f(t)$, and this spectrum is purely absolutely continuous in Λ_0 . Moreover, the density of the spectral measure constructed with reading point 0 is bounded above and bounded below away from zero on any compact subset of Λ_0 .*

Proof. Here Λ_0 is contained in the spectrum of the unique self-adjoint extension of H , which we denote again by H , because of Lemma II.1. In order to prove that the spectrum of H is purely absolutely continuous in Λ_0 , we show that the restriction of its spectral measure σ to Λ_0 is absolutely continuous with respect to Lebesgue's measure in Λ_0 , and without any loss of generality we may as well assume that Λ_0 is a bounded interval.

We claim that for each open interval I whose closure is contained in Λ_0 , there exists a constant $c = c(I) > 0$ such that

$$\sigma(A) \leq c |A| \quad (\text{II.24})$$

for all open subintervals A of I , where $|A|$ denotes the Lebesgue measure of A .

We first check that our claim implies the desired conclusion on absolute continuity. Let us assume the existence of a Borel set A contained in Λ_0 such that $|A| = 0$ and $\sigma(A) > 0$. By the inner regularity of the measure σ , one can find a compact set K such that $K \subset A$ (and so $|K| = 0$) and $\sigma(K) > 0$. By the outer regularity of Lebesgue's measure for each integer n we can find an open set O_n containing K , whose closure is contained in Λ_0 and whose Lebesgue's measure is less than $1/n$. Each O_n can be written as the union of an at most countable family of disjoint open intervals. Hence, for each n , we can assume that K is contained in the union of a finite family of disjoint open intervals, say $I_{n,k}$, whose closures are contained in Λ_0 and such that

$$\sum_k |I_{n,k}| \leq \frac{1}{n}.$$

Using our claim (II.24) with $I = O$ (note that without any loss of generality we may assume that all the O_n 's are contained in O_1) and $A = I_{n,k}$, we obtain

$$\sigma(K) \leq \sum_k \sigma(I_{n,k}) \leq c \sum_k |I_{n,k}| \leq c |O_n| \leq \frac{c}{n}$$

for all n and a constant c independent of n . This proves $\sigma(K) = 0$ which makes our assumption on A impossible.

Next we prove our claim. Since we are in the Weil limit point case near $+\infty$, the spectral measure σ is the limit (in the sense of vague convergence of measures) of the spectral measures $\sigma_{-L,L,\alpha,\beta}$ as $L > 0$ tends to $+\infty$, irrespectively of the boundary conditions α and β at $-L$ and L , respectively. Because of the vague convergence for each open A we have

$$\sigma(A) \leq \liminf_{L \rightarrow \infty} \sigma_{-L,L,\alpha,\beta}(A) \quad (\text{II.25})$$

for all α and β . Consequently, using Fatou's lemma and Lemma II.2 we obtain

$$\begin{aligned} \sigma(A) &= \frac{1}{\pi} \int_{-\pi/2}^{+\pi/2} \sigma(A) d\alpha \\ &\leq \liminf_{L \rightarrow \infty} \frac{1}{\pi} \int_{-\pi/2}^{+\pi/2} \sigma_{-L,L,\alpha,\beta}(A) d\alpha \\ &= \liminf_{L \rightarrow \infty} \frac{1}{\pi} \int_A r_\lambda(-L, \theta_{\lambda,L}(0, \beta))^{-2} d\lambda, \end{aligned} \quad (\text{II.26})$$

where $r_\lambda(t, \theta)$ denotes the amplitude at t of the solution of $-y'' + (q - \lambda)y = 0$ whose amplitude and phase at the origin are 1 and θ , respectively. Hence, if Θ is a unit vector in \mathbb{R}^2 with phase $\theta_{\lambda,L}(0, \beta)$, the integrand in (II.26) can be estimated by

$$\begin{aligned} r_\lambda(-L, \theta_{\lambda,L}(0, \beta))^{-2} &= \|U_\lambda(-L, 0) \Theta\|^{-2} \leq \|U_\lambda(0, -L)\|^2 \\ &\leq \|U_\lambda(-L, 0)\|^2 \\ &\leq \left(\sup_{t \leq 0} \|U_\lambda(t, 0)\| \right)^2. \end{aligned} \quad (\text{II.27})$$

Now, if I is any open interval whose closure is contained in A_0 , by our assumption of local uniform stability, there exists a finite constant $k = k(I)$ such that

$$\sup_{t \leq 0, \lambda \in I} \|U_\lambda(t, 0)\| \leq k \quad (\text{II.28})$$

and consequently, for all open subintervals A of I the conjunction of (II.26)–(II.28) gives

$$\sigma(A) \leq \frac{k^2}{\pi} |A| \quad (\text{II.29})$$

and this completes the proof of our claim.

Now, if A is a compact set contained in A_0 , because of the vague convergence, instead of (II.25) we have

$$\sigma(A) \geq \limsup_{L \rightarrow \infty} \sigma_{-L, L, \alpha, \beta}(A)$$

and (II.26) becomes

$$\sigma(A) \geq \limsup_{L \rightarrow \infty} \frac{1}{\pi} \int_A r_\lambda(-L, \theta_{\lambda, L}(0, \beta))^{-2} d\lambda \quad (\text{II.30})$$

but with the same notation as above, the integrand in (II.30) can be estimated as

$$r_\lambda(-L, \theta_{\lambda, L}(0, \beta))^{-2} = \|U_\lambda(-L, 0) \Theta\|^{-2} \geq \left(\sup_{t \leq 0} \|U_\lambda(t, 0)\| \right)^{-2}$$

so that (II.29) becomes

$$\sigma(A) \geq \frac{k^{-2}}{\pi} |A| \quad (\text{II.31})$$

and the conclusion on the density of σ with respect to Lebesgue's measure follows from the conjunction of (II.29) and (II.31). ■

As immediate corollaries of Theorem II.4 we have the results announced in the introduction.

COROLLARY II.5. *Let $q(t)$ be a real-valued locally integrable function which is bounded below near $-\infty$ and $+\infty$ by a function of the form $-a_0(t^2 + 1)$ for some $a_0 > 0$ and which is integrable near $-\infty$ or $+\infty$. Then the spectrum of the unique self-adjoint extension of the operator H defined for $f \in C_0^\infty(\mathbb{R})$ by $[Hf](t) = -f''(t) + q(t)f(t)$, contains $(0, \infty)$ and is purely absolutely continuous there.*

Proof. It is easy to check that the propagator of the vector differential equation (II.20) with $q = 0$ satisfies for each $\lambda > 0$

$$\sup_{t, s \in \mathbb{R}} \|U_\lambda(t, s)\| \leq 2 \max(\lambda, \lambda^{-1}). \quad (\text{II.32})$$

Let us assume that q is integrable near $+\infty$, for example. Classical arguments on the stability of ordinary differential equations under small perturbations (see, for example, [3, 5]) lead from (II.32) to

$$\sup_{t, s \geq 0} \|U_\lambda(t, s)\| \leq 2 \max(\lambda, \lambda^{-1}) e^{2 \max(\lambda, \lambda^{-1}) \int_0^\infty |q(t)| dt}$$

which shows that $(0, \infty)$ is a set of local uniform stability for (II.20) and we conclude by using Theorem II.4. ■

COROLLARY II.6. *Let $q_1(t)$ be a real-valued bounded periodic function on \mathbb{R} and let $\bigcup_i [a_i, b_i]$ with $b_i < a_{i+1}$ be the spectrum of the unique self-adjoint extension of the operator H_1 defined for $f \in C_0^\infty(\mathbb{R})$ by $[H_1 f](t) = -f''(t) + q_1(t)f(t)$. If $q_2(t)$ is any real-valued locally integrable function which is bounded below near $+\infty$ by a function of the form $-a_0(t^2 + 1)$ for some $a_0 > 0$ and if the function $q(t)$ is defined by*

$$\begin{aligned} q(t) &= q_1(t), & \text{if } t \leq 0, \\ &= q_2(t), & \text{if } t > 0, \end{aligned}$$

then the spectrum of the unique self-adjoint extension of the operator H defined for $f \in C_0^\infty(\mathbb{R})$ by $[Hf](t) = -f''(t) + q(t)f(t)$ contains $\bigcup_i (a_i, b_i)$ and is purely absolutely continuous there.

Proof. It is an immediate consequence of Theorem II.4 above and the well-known fact that the (a_i, b_i) are sets of local uniform stability in the sense of Definition II.3 (see, for example, [3, Chap. II; 10]). ■

The application which is described in Corollary II.8 below has been pointed out to us by Barry Simon.

DEFINITION II.7. *A real-valued function $q(t)$ is said to be of the Dinaburg-Sinai type near $+\infty$ (resp. $-\infty$) if it admits a representation of the form*

$$q(t) = \sum_{k_1, \dots, k_n} c_{k_1, \dots, k_n} e^{i(k_1 \omega_1 + \dots + k_n \omega_n)t},$$

where the so-called frequencies $\omega_1, \dots, \omega_n$ are given and the so-called Fourier coefficients $c_k = c_{k_1, \dots, k_n}$ satisfy $\sum_k |c_k|^2 < +\infty$ (i.e., $q(t)$ is quasi-periodic) and are such that the following two conditions are satisfied:

(A₁) *there exists a constant $C_1 > 0$ such that for all multiintegers $k = (k_1, \dots, k_n)$ such that $|k| = \sum_{j=1}^n |k_j| > 0$ we have*

$$\left| \sum_{j=1}^n k_j \omega_j \right| \geq C_1 |k|^{-n-1},$$

(A₂) *there exist positive constants C_2 and ρ such that the Fourier coefficients c_k satisfy*

$$|c_k| \leq C_2 e^{-\rho |k|}.$$

The first theorem of [8] asserts that if $q(t)$ is of the Dinaburg–Sinai type near $+\infty$ (resp. $-\infty$) then, for each $\varepsilon > 0$ there exist a constant $C_3 = C_3(\varepsilon)$, real numbers λ_k for $k \in \mathbb{Z}^n$, and λ^0 sufficiently large such that for λ in the set

$$S = \left\{ \lambda \geq \lambda^0; |\sqrt{\lambda} - \lambda_k| > C_3 \exp \left[-\frac{\|k\|}{\log^{1+a} \|k\|} \right], k \in \mathbb{Z}^n \right\},$$

there exists a fundamental system, say $\{y_{\lambda,1}(t), y_{\lambda,2}(t)\}$, of solutions of Eq. (II.20) for t in this neighborhood of $+\infty$ (resp. $-\infty$) such that $y_{\lambda,2}(t) = y_{\lambda,1}(t)$ and $y_{\lambda,1}(t) = \chi_\lambda(t) e^{ia(\lambda)t}$, where $a(\lambda) \in \mathbb{R}$ and $\chi_\lambda(t)$ is a quasi-periodic function with $\omega = (\omega_1, \dots, \omega_n)$ as independent frequencies.

It turns out that the function $a(\lambda)$ satisfies

$$|a(\lambda) - \lambda^{1/2}| \leq C_4 \lambda^{-1/2}$$

for some constant $C_4 = C_4(\varepsilon)$ so that $a(\lambda)$ is locally bounded. Moreover $\chi_\lambda(t)$ and its derivative are bounded in t , uniformly in λ restricted to bounded subsets of S as an inspection of the proof shows. Consequently, estimate (II.28) holds for all bounded subsets \bar{I} of S and the proof of Theorem II.4 gives the following strengthening of the second theorem of [8]:

COROLLARY II.8. *Let $q(t)$ be a real-valued locally integrable function on \mathbb{R} which is bounded below near $+\infty$ and $-\infty$ by a function of the form $-a_0(t^2 + 1)$ for some constant $a_0 > 0$ and which is of the Dinaburg–Sinai type near $+\infty$ or $-\infty$. Then the set S defined above is in the spectrum of the unique self-adjoint extension in $L^2(\mathbb{R})$ of the operator H defined for $f \in C_0^\infty(\mathbb{R})$ by $[Hf](t) = -f''(t) + q(t)f(t)$. This spectrum is purely absolutely continuous in the interior of S and the density of the spectral measure constructed with reading point 0 is bounded above and bounded below away from zero on any bounded subset of the interior of S . In any case (and especially if S has empty interior) this spectral measure has an absolute component on S the density of which is locally bounded away from zero.*

Our assumption of local uniform stability is of course not necessary for the spectrum to be absolutely continuous. Theorem II.9 is some sort of generalization of Theorem II.4. Corollary II.10 shows its usefulness in the study of some unstable cases. It is stated separately because the checking of its assumptions is usually more involved and this could have obscured the simplicity of the idea and the proof leading to our deterministic results.

THEOREM II.9. *Let $q(t)$ be a real-valued locally integrable function on \mathbb{R} which is bounded below near $+\infty$ and $-\infty$ by a function of the form $-a_0(t^2 + 1)$ for t for some constant $a_0 > 0$. Let A_0 be an open subset of \mathbb{R}*

such that, for each compact set I contained in Λ_0 there exist a constant $k > 0$ and a sequence of probability measures $\{\mu_n; n \geq 1\}$ such that

$$\forall a \in \mathbb{R} \quad \lim_{n \rightarrow \infty} \mu_n([a, \infty)) = 0 \quad (\text{II.33})$$

and such that

$$\int [|y(t)|^2 + |y'(t)|^2]^{-1} d\mu_n(t) \leq k \quad (\text{II.34})$$

for all integers n and all functions $y(t)$ satisfying $|y(0)|^2 + |y'(0)|^2 = 1$ and

$$-y''(t) + (q(t) - \lambda)y(t) = 0, \quad t \in \mathbb{R}, \quad (\text{II.35})$$

for some $\lambda \in I$. Then the part in Λ_0 , if any, of the spectrum of the unique self-adjoint extension in $L^2(\mathbb{R})$ of the operator H defined for $f \in C_0^\infty(\mathbb{R})$ by $[Hf](t) = -f''(t) + q(t)f(t)$, is purely absolutely continuous and the density of the spectral measure constructed with reading point 0 is bounded on each compact subset of Λ_0 .

Proof. As argued in the first part of the proof of Theorem II.4 we need only proving (II.24). Our assumption (II.33) implies the existence of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sequence $\{L_n; n \geq 1\}$ of random variables such that

$$\mathbb{P}\{L_n \in dt\} = \mu_n, \quad n \geq 1, \quad (\text{II.36})$$

and

$$\lim_{n \rightarrow \infty} L_n(\omega) = -\infty \quad \text{for } \mathbb{P}\text{-almost all } \omega \in \Omega. \quad (\text{II.37})$$

As before (recall (II.25)) we have

$$\sigma(A) \leq \liminf_{n \rightarrow \infty} \sigma_{L_n(\omega), n, \alpha, \beta}(A)$$

for all α and β and for \mathbb{P} -almost all $\omega \in \Omega$ because of (II.37), so that again using Fatou's lemma we obtain

$$\begin{aligned} \sigma(A) &= \frac{1}{\pi} \int_{\Omega} \int_{-\pi/2}^{+\pi/2} \sigma(A) d\alpha d\mathbb{P}(\omega) \\ &\leq \frac{1}{\pi} \liminf_{n \rightarrow \infty} \int_{\Omega} \int_{-\pi/2}^{+\pi/2} \sigma_{L_n(\omega), n, \alpha, \beta}(A) d\alpha d\mathbb{P}(\omega) \\ &= \frac{1}{\pi} \liminf_{n \rightarrow \infty} \int_{\Omega} \int_{\Lambda} r_{\lambda}(L_n(\omega), \theta_{\lambda, n}(0, \beta))^{-2} d\lambda d\mathbb{P}(\omega) \end{aligned}$$

using once more our crucial Lemma II.2 and the same notations as before. Now, by Fubini's theorem and (II.37) we obtain

$$\sigma(A) \leq \frac{1}{\pi} \liminf_{n \rightarrow \infty} \int_A \int r_\lambda(t, \theta_{\lambda,n}(0, \beta))^{-2} d\mu_n(t) d\lambda \leq \frac{k}{\pi} |A|$$

by our assumption (II.34), and this completes the proof. ■

We apply the above result to one-dimensional Schrödinger operators with a constant electric field. This application precises and reinforces the deterministic result of [1].

COROLLARY II.10. *Let $q(t)$ be a real-valued locally integrable function on \mathbb{R} which satisfies*

$$q(t) \geq -a_0(t^2 + 1), \quad t \geq 0 \quad (\text{II.38})$$

for some constant $a_0 > 0$, and

$$q(t) = -F|t|^\alpha + v(t), \quad t < 0, \quad (\text{II.39})$$

where $\alpha \in [1, 2]$, $F > 0$ and $v(t)$ is twice continuously differentiable function which is bounded and has bounded derivatives. Then the spectrum of the unique self-adjoint extension in $L^2(\mathbb{R})$ of the operator H defined for $f \in C_0^\infty(\mathbb{R})$ by $[Hf](t) = -f''(t) + q(t)f(t)$ is the whole real line, it is purely absolutely continuous and the density of the spectral measure constructed with reading point 0 is locally bounded.

The proof relies on the technical Lemma II.11. All the steps of the first part of its proof are well known from the theory of ordinary differential equations. Nevertheless we give a complete proof because our estimate (II.40) depends on a careful analysis of the influences the initial conditions and the parameter λ have on the asymptotic formulas which are known or expected to hold in such a situation.

LEMMA II.11. *Let the function $q(t)$ be as in the statement of Corollary II.10. Then for any bounded interval A of \mathbb{R} and any constant $K > 1$, there exist a real $t_1 < 0$ and a constant $K' > 0$ such that*

$$\frac{1}{b-a} \int_a^b [|y(t)|^2 + |y'(t)|^2]^{-1} dt \leq K' \quad (\text{II.40})$$

for all a and b such that $a < b < t_1$, $|a/b|^\alpha \leq K$ and $b-a > K^{-1}$, any solution $y(t)$ of (II.35) such that $|y(0)|^2 + |y'(0)|^2 = 1$, and any $\lambda \in A$.

Proof. To simplify our notations and without any loss of generality we

prove the analog result when t is replaced by $-t$ in both the assumptions and the conclusion. So we should think that $t \rightarrow \infty$ instead of $t \rightarrow -\infty$. We fix a closed bounded interval A in \mathbb{R} and we restrict λ to vary in A . We also fix $t_0 > 0$ such that $\lambda - q(t) \geq 1$ whenever $\lambda \in A$ and $t \geq t_0$. For each $\lambda \in A$, the map s_λ defined by

$$[t_0, \infty) \ni t \rightarrow s_\lambda(t) = \int_{t_0}^t [\lambda - q(u)]^{1/2} du \in [0, \infty) \quad (\text{II.41})$$

is one-to-one, onto, continuously differentiable and its derivative $s'_\lambda(t) = [\lambda - q(t)]^{1/2}$ is positive. Whenever y is a twice continuously differentiable function on $[t_0, \infty)$ we define the function z on $[0, \infty)$ by

$$z(u) = [\lambda - q(s_\lambda^{-1}(u))]^{1/4} y(s_\lambda^{-1}(u)) \quad \text{for } u \geq 0$$

or equivalently by

$$y(t) = [\lambda - q(t)]^{-1/4} z(s_\lambda(t)) \quad \text{for } t \geq t_0. \quad (\text{II.42})$$

The function z is twice continuously differentiable and y is a solution of

$$-y''(t) + (q(t) - \lambda)y(t) = 0 \quad (\text{II.43})$$

on $[t_0, \infty)$ if and only if z is a solution of

$$-z''(s) + (-1 + b_\lambda(s))z(s) = 0 \quad (\text{II.44})$$

on $[0, \infty)$, where

$$b_\lambda(s) = \left(-\frac{1}{4}q''(t)[\lambda - q(t)]^{-2} - \frac{5}{16}q'(t)^2[\lambda - q(t)]^{-3}\right) \Big|_{t=s_\lambda^{-1}(s)}. \quad (\text{II.45})$$

First we note that

$$\int_0^\infty |b_\lambda(s)| ds < +\infty. \quad (\text{II.46})$$

Next we remark that $|b_\lambda(s)| \leq c_1[s_\lambda^{-1}(s)]^{-2\alpha}$ and $s_\lambda(t) \leq c_2 t^{1+\alpha/2}$ for some positive constants c_1 and c_2 so that we have

$$|b_\lambda(s)| \leq \beta(s) \quad (\text{II.47})$$

with $\beta(s) = (c_1 c_2^{2/(2+\alpha)}) s^{-4\alpha/(2+\alpha)}$. Note that $\beta(s)$ is independent of $\lambda \in A$ and that we can choose $s_0 \geq 0$ so that

$$\delta = \int_{s_0}^\infty \beta(s) ds < 1. \quad (\text{II.48})$$

Now, in order to solve (II.44) we consider the first order vector differential equation

$$Z'(s) = [A + B_\lambda(s)] Z(s), \quad (\text{II.49})$$

where

$$Z(s) = \begin{bmatrix} z(s) \\ z'(s) \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \quad \text{and} \quad B_\lambda(s) = \begin{bmatrix} 0 & 0 \\ b_\lambda(s) & 0 \end{bmatrix},$$

because we want to control the asymptotic behavior of both $z(s)$ and $z'(s)$. For each element $Z(s)$ of the Banach space \mathcal{Z} for the norm

$$\|Z\|_\infty = \sup_{s \geq s_0} \|Z(s)\|$$

of bounded continuous functions on $[s_0, \infty)$ taking values in \mathbb{R}^2 we set

$$[S_\lambda Z](s) = - \int_s^\infty e^{(s-u)A} B_\lambda(u) Z(u) du, \quad s \geq s_0. \quad (\text{II.50})$$

S_λ so defined is a bounded operator on \mathcal{Z} . Moreover its norm is less than δ because of (II.47) and (II.48). Consequently, $I - S_\lambda$ is invertible (I denotes the identity operator of \mathcal{Z}). We also notice that if X and Z are two elements of \mathcal{Z} related by

$$Z = (I - S_\lambda)^{-1} X,$$

then Z is a solution of (II.49) if and only if X is a solution of

$$X'(s) = AX(s), \quad s \geq s_0 \quad (\text{II.51})$$

and moreover

$$\|X(s) - Z(s)\| = \|(S_\lambda Z)(s)\| \leq \left(\int_s^\infty \|B_\lambda(u)\| du \right) \|Z\|_\infty. \quad (\text{II.52})$$

If we choose

$$X_\pm(s) = e^{\pm is} \begin{bmatrix} 1 \\ \pm i \end{bmatrix} \quad \text{for } s \geq s_0,$$

then $X_\pm \in \mathcal{Z}$ and X_\pm is a solution of (II.51). From this we deduce that $Z_{\lambda, \pm} = (I - S_\lambda)^{-1} X_\pm$ is a bounded solution of (II.49) and that

$$\|Z_{\lambda, \pm}(s) - X_\pm(s)\| \leq 2^{1/2} (1 - \delta)^{-1} \int_s^\infty \beta(u) du \quad (\text{II.53})$$

because of (II.52) and (II.47) and because

$$\|Z_{\lambda,\pm}\|_{\infty} \leq \| (I - S_{\lambda})^{-1} \| \|X_{\pm}\|_{\infty} \leq (1 - \delta)^{-1} \sqrt{2}.$$

Moreover, expanding $(I - S_{\lambda})^{-1}$ in an uniformly convergent series (recall (II.48)) it is easy to check that for each fixed $s \geq s_0$, $Z_{\lambda,\pm}(s)$ is a continuous function of $\lambda \in \mathcal{A}$ by Lebesgue's dominated convergence theorem.

If we substitute $s_{\lambda}(t)$ to s in (II.53) and if we use (II.42) we obtain a basis $\{y_{\lambda,+}(t), y_{\lambda,-}(t)\}$ for the space of solutions of (II.43) which satisfies

$$\begin{aligned} |y_{\lambda,\pm}(t) - [\lambda - q(t)]^{-1/4} e^{\pm i s_{\lambda}(t)}| &\leq 2^{1/2} (1 - \delta)^{-1} [\lambda - q(t)]^{-1/4} \int_{s_{\lambda}(t)}^{\infty} \beta(u) du \\ |y'_{\lambda,\pm}(t) - \pm i [\lambda - q(t)]^{1/4} e^{\pm i s_{\lambda}(t)}| &\leq \frac{1}{4} |q'(t)| [\lambda - q(t)]^{-1/4} + 2^{1/2} (1 - \delta)^{-1} \\ &\quad \times (1 + [\lambda - q(t)]^{1/4}) \int_{s_{\lambda}(t)}^{\infty} \beta(u) du \end{aligned}$$

and which are continuous in λ for each fixed t , as well as their derivatives. Consequently we have

$$y_{\lambda,\pm}(t) = [\lambda - q(t)]^{-1/4} e^{\pm i s_{\lambda}(t)} + O(t^{1-7\alpha/4}) \quad (\text{II.54})$$

$$y'_{\lambda,\pm}(t) = \pm i [\lambda - q(t)]^{1/4} e^{\pm i s_{\lambda}(t)} + O(t^{1-5\alpha/4}), \quad (\text{II.55})$$

where $O(t^{\beta})$ stands for any function of t whose modulus or absolute value is bounded above by a constant independent of $\lambda \in \mathcal{A}$ times t^{β} .

To prove our claim (II.40) let us pick a solution $y(t)$ of (II.42) such that $y(0)^2 + y'(0)^2 = 1$. Then

$$y(t) = \alpha_1 y_{\lambda,1}(t) + \alpha_2 y_{\lambda,2}(t)$$

for some numbers α_1 and α_2 , where $y_{\lambda,1}(t) = \frac{1}{2}(y_{\lambda,+}(t) + y_{\lambda,-}(t))$ and $y_{\lambda,2}(t) = (1/2i)(y_{\lambda,+}(t) - y_{\lambda,-}(t))$. The continuity of $y_{\lambda,\pm}(t)$ in λ for fixed t implies the existence of positive constants k_1 and k_2 independent of such $y(t)$'s such that

$$0 < k_1 \leq \alpha_1^2 + \alpha_2^2 \leq k_2 < +\infty. \quad (\text{II.56})$$

and thus

$$\begin{aligned} y(t)^2 + y'(t)^2 &= ((\alpha_1 \cos s_{\lambda}(t) + \alpha_2 \sin s_{\lambda}(t))[\lambda - q(t)]^{-1/4} + O(t^{1-7\alpha/4}))^2 \\ &\quad + ((-\alpha_1 \sin s_{\lambda}(t) + \alpha_2 \cos s_{\lambda}(t))[\lambda - q(t)]^{1/4} + O(t^{1-5\alpha/4}))^2 \\ &= (\alpha_1^2 + \alpha_2^2)[([\lambda - q(t)]^{-1/4} \cos s_{\lambda}(t) - \theta) + O(t^{1-7\alpha/4})]^2 \\ &\quad + ([\lambda - q(t)]^{1/4} \sin(-s_{\lambda}(t) + \theta) + O(t^{1-5\alpha/4}))^2, \end{aligned}$$

where $\theta \in [0, 2\pi)$ has been chosen so that $(\alpha_1^2 + \alpha_2^2)^{1/2} \cos \theta = \alpha_1$ and $(\alpha_1^2 + \alpha_2^2)^{1/2} \sin \theta = \alpha_2$. Using (II.56) we obtain

$$\begin{aligned} y(t)^2 + y'(t)^2 &\geq k_1(|\lambda - q(t)|^{-1/2} \cos^2(s_\lambda(t) - \theta) \\ &\quad + (-[\lambda - q(t)]^{1/4} \sin(s_\lambda(t) - \theta) \\ &\quad + O(t^{1-5\alpha/4})) + O(t^{1-2\alpha})). \end{aligned} \quad (\text{II.57})$$

Now we pick a and b such that $t_1 \leq a < b$, $b^\alpha \leq Ka^\alpha$, and $b - a > K^{-1}$. To prove (II.40) we first perform the change of variable $u = s_\lambda(t) - \theta$

$$\begin{aligned} \frac{1}{b-a} \int_a^b [y(t)^2 + y'(t)^2]^{-1} dt &= \frac{1}{b-a} \int_{s_\lambda(a)-\theta}^{s_\lambda(b)-\theta} [y(s_\lambda^{-1}(u+\theta))^2 \\ &\quad + y'(s_\lambda^{-1}(u+\theta))^2]^{-1} [\lambda - q(s_\lambda^{-1}(u+\theta))]^{-1/2} du \\ &\leq \frac{1}{k_1(b-a)} \int_{s_\lambda(a)-\theta}^{s_\lambda(b)-\theta} \Phi(u) du \end{aligned}$$

if we set for notation convenience

$$\begin{aligned} \Phi(u) &= [\cos^2 u + (-[\lambda - q(s_\lambda^{-1}(u+\theta))]^{1/2} \sin u \\ &\quad + O(s_\lambda^{-1}(u+\theta)^{1-\alpha}))^2 \\ &\quad + O(s_\lambda^{-1}(u+\theta)^{1-3\alpha/2})]^{-1}. \end{aligned}$$

Let $c > 0$ be an upper bound for the absolute value of the function $O(s_\lambda^{-1}(u+\theta)^{1-\alpha})$ and let us set

$$A_1 = \{u \in [s_\lambda(a) - \theta, s_\lambda(b) - \theta]; [\lambda - q(s_\lambda^{-1}(u+\theta))]^{1/2} |\sin u| \geq 2c\}$$

and

$$A_2 = [s_\lambda(a) - \theta, s_\lambda(b) - \theta] \setminus A_1.$$

We can also assume without any loss of generality that t_1 has been chosen large enough so that the absolute value of the function $O(s_\lambda^{-1}(u+\theta)^{1-3\alpha/2})$ is bounded by $\frac{1}{4}$. Now we have

$$\begin{aligned} \int_{A_1} \Phi(u) du &\leq 4 \int_{A_1} \frac{du}{1 + [\lambda - 4 - q(s_\lambda^{-1}(u+\theta))] \sin^2 u} \\ &\leq 4 \int_{A_1} \frac{du}{1 + \tilde{q} \sin^2 u} \end{aligned}$$

where $\tilde{q} = \inf\{\lambda - 4 - q(t); \lambda \in \Lambda, t \in [a, b]\}$

$$\begin{aligned} &\leq 4 \frac{s_\lambda(b) - s_\lambda(a)}{[1 + \tilde{q}]^{1/2}} \\ &\leq k_3(b - a), \end{aligned}$$

where k_3 is not only independent of $\lambda \in \Lambda$ but also of a and b because $(b/a)^\alpha$ is bounded by K and $b - a$ by K^{-1} . Furthermore, we notice that $u \in A_2$ implies

$$\Phi(u) \geq \cos^2 u + O(s_\lambda^{-1}(u + \theta)^{1-3\alpha/2}) \geq \frac{1}{4}$$

because

$$\cos^2 u \geq 1 - (c^2/4)[\lambda - q(s_\lambda^{-1}(u + \theta))]^{-1} \geq \frac{1}{2}$$

provided t_1 has been chosen large enough so that $q(t) \leq \lambda - c^2/2$ for $t \geq t_1$ and $\lambda \in \Lambda$. Consequently,

$$\int_{A_2} \Phi(u) du \leq 4 |\{u \in [s_\lambda(a) - \theta, s_\lambda(b) - \theta]; |\sin u| \leq \bar{q}\}|$$

where $\bar{q} = \sup\{c/2[\lambda - q(t)]^{1/2}; \lambda \in \Lambda, t \in [a, b]\}$

$$\leq \frac{8}{\pi} \bar{q} [s_\lambda(b) - s_\lambda(a)] \leq k_4(b - a)$$

as before and this completes the proof. ■

Remark II.12. Estimates (II.54) and (II.55) on the fundamental system $\{y_{\lambda,+}(t), y_{\lambda,-}(t)\}$ of solutions of (II.35) show that this equation cannot have nontrivial sets of local uniform stability because they imply

$$\lim_{t \rightarrow \infty} t^{-\alpha/2} \sup_{0 \leq s \leq t} \|U_\lambda(0, s)\| = \infty.$$

Proof of Corollary II.10. The spectrum of H is the whole real line because for each $\lambda \in \mathbb{R}$ an easy computation (see, for example, [9, p. 1413; 3, Chap. I]) shows that $-y'' + (q - \lambda)y = 0$ cannot have square integrable solutions near $-\infty$ so that, since H is essentially self-adjoint λ belong to the essential spectrum, so to the spectrum of H . Moreover the Lemma I.11 above show that the assumptions of Theorem II.9 are satisfied by any bounded interval A_0 in \mathbb{R} . ■

III. THE RANDOM CASE

The main results of this section are Theorems III.4 and III.5. Their proofs have in common what we regard as the basic method to prove the existence of point spectrum with exponentially decaying eigenfunctions in the one-dimensional case. This method is explained in full detail in [2] and it has been successfully used in [1, 2]. To avoid going through its various steps twice we prefer presenting them once in Lemma III.2, even though its formulation may look lengthy and technical at first sight. But first we prove an elementary analytical result in order to enlighten our crucial assumption (III.4).

LEMMA III.1. *Let φ be a locally bounded function such that*

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log |\varphi(t)| \leq 0, \quad (\text{III.1})$$

and let $r(t)$ be any nonnegative solution of $r'(t) = \varphi(t) r(t)$. Then, if there exist $\alpha > 0$ and $\delta > 0$ such that

$$\int_0^\infty r(t)^\delta e^{\delta \alpha t} dt < +\infty, \quad (\text{III.2})$$

then for each $0 < \varepsilon < \alpha$ there exists a constant $k_\varepsilon > 0$ such that

$$r(t) \leq k_\varepsilon e^{-(\alpha - \varepsilon)t}, \quad t \geq 0. \quad (\text{III.3})$$

Proof. For each $T > 0$, a simple integration by parts gives

$$\begin{aligned} \int_0^T r(t)^\delta e^{\delta(\alpha - \varepsilon)t} dt &= \frac{r(T)^\delta}{\delta(\alpha - \varepsilon)} e^{\delta(\alpha - \varepsilon)T} - \frac{r(0)^\delta}{\delta(\alpha - \varepsilon)} \\ &\quad - \frac{1}{\alpha - \varepsilon} \int_0^T \varphi(t) r(t)^\delta e^{\delta(\alpha - \varepsilon)t} dt. \end{aligned}$$

The first integral converges at $T \rightarrow \infty$ by our assumption (III.2). Moreover assumption (III.1) implies that $|\varphi(t)| \leq k' e^{\delta \varepsilon t}$ for some constant k' independent of $t \geq 0$. Hence the second integral converges also as $T \rightarrow \infty$ by the same assumption (III.2). Consequently, $r(T) e^{(\alpha - \varepsilon)T}$ tends to a finite limit as $T \rightarrow \infty$ and this implies (III.3). ■

Throughout this section, all the stochastic processes and the random variables to be considered will be implicitly assumed to be defined on the same complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and \mathbb{E} will denote expectation with respect to the measure \mathbb{P} .

Moreover $\{q(t, \omega); t \in \mathbb{R}, \omega \in \Omega\}$ will always be a real-valued measurable stochastic process such that for \mathbb{P} -almost all $\omega \in \Omega$ the function $q(t, \omega)$ is locally bounded and bounded (near plus and minus infinity) below by a function of the form $-a_0(t^2 + 1)$ for some constant $a_0 > 0$ depending possibly on $\omega \in \Omega$ and above by a polynomial function of t which can also depend on $\omega \in \Omega$. Then for \mathbb{P} -almost all $\omega \in \Omega$, the formal symmetric operator $H(\omega) = -(d^2/dt^2) + q(t, \omega)$ can actually be defined on $C_0^\infty(\mathbb{R})$ and is in the limit point case at both plus and minus infinity. We will use the same notation, namely $H(\omega)$, for its unique self-adjoint extension in $L^2(\mathbb{R})$, and we will denote by σ^ω its spectral measure constructed with reading point 0. It is easy to see that σ^ω is measurable as a function of $\omega \in \Omega$ (see, for example, [3, Chap. V]).

LEMMA III.2. *Let $\Lambda \subset \mathbb{R}$ be an open interval and for each $\lambda \in \Lambda$ and $\omega \in \Omega$ we assume that there exist positive numbers $\alpha_+(\lambda, \omega)$ and $\alpha_-(\lambda, \omega)$ and a unit vector $\Theta_{\lambda, \omega}^-$ in \mathbb{R}^2 such that*

$$(i) \quad \lim_{t \rightarrow -\infty} \frac{1}{t} \log \|U_\lambda(t, 0) \Theta_{\lambda, \omega}^-\| = \alpha_-(\lambda, \omega)$$

and

$$\limsup_{t \rightarrow -\infty} \frac{1}{t} \log \|U_\lambda(t, 0) \Theta\| < 0$$

whenever $\Theta \in \mathbb{R}^2$ is not colinear to $\Theta_{\lambda, \omega}^-$;

(ii) α_+ is a measurable function of (λ, ω) in $\Lambda \times \Omega$ and $\inf_{\lambda \in \Lambda, \omega \in \Omega} \alpha_+(\lambda, \omega) > 0$;

(iii) for each $\varepsilon > 0$ there exist $\delta > 0$ and $\tau \geq 0$ such that

$$\mathbb{E} \left\{ \int_\Lambda \sigma^\omega(d\lambda) \left(\inf_{\|\Theta\|=1} \int_\tau^\infty \|U_\lambda(t, \tau) \Theta\|^\delta e^{\delta[\alpha_+(\lambda, \omega) - \varepsilon]t} dt \right) \right\} < \infty. \quad (\text{III.4})$$

Then, for \mathbb{P} -almost all $\omega \in \Omega$, the spectrum of $H(\omega)$ in Λ (if any) is pure point with exponentially decaying eigenfunctions, the rate of exponential fall off of an eigenfunction corresponding to an eigenvalue $\lambda \in \Lambda$ (if any) being given by $\alpha_-(\lambda, \omega)$ for $t \rightarrow -\infty$ and bounded below by $\alpha_+(\lambda, \omega)$ for $t \rightarrow +\infty$.

Here, as in the preceding section $U_\lambda(t, \tau)$ denotes the propagator of the vector differential equation

$$Y'(t) = \begin{bmatrix} 0 & 1 \\ q(t, \omega) - \lambda & 0 \end{bmatrix} Y(t).$$

It is a two-by-two unimodular random matrix but we will not make explicit its dependence on $\omega \in \Omega$.

Remark III.3. The appearance of $\tau \geq 0$ in (III.4) is for later convenience. In fact τ could be (and will be) random (i.e., depending on $\omega \in \Omega$) without affecting the conclusion. In most applications (see, for example, [2, Sect. II]) condition (ii) is automatically satisfied. Indeed $\alpha_+(\lambda, \omega)$ happens to be the upper Ljapunov exponent of a product of matrices taken from an ergodic sequence. The ergodicity makes it independent of $\omega \in \Omega$, a Furstenberg-type argument gives its positivity and the full checking of (ii) is completed by proving its continuity in $\lambda \in A$. Moreover, [6] gives its subharmonicity as a function of λ and this implies that $\alpha_+(\lambda)$ is the exact rate of exponential fall off and not merely a lower bound as stated in Lemma III.2 (see [6] for a proof).

Proof of Lemma III.2. Let us fix $\varepsilon > 0$ such that $\varepsilon < \inf_{\lambda \in A, \omega \in \Omega} \alpha_+(\lambda, \omega)$ and let us apply assumption (iii) to $\varepsilon/2$. For some $\delta > 0$ and some $\tau \geq 0$ we know that for \mathbb{P} -almost all $\omega \in \Omega$ and σ^ω -almost all $\lambda \in A$

$$\inf_{\|\Theta\|=1} \int_{\tau}^{\infty} \|U_{\lambda}(t, \tau) \Theta\|^{\delta} e^{\delta[\alpha_+(\lambda, \omega) - \varepsilon/2]t} dt < +\infty$$

from which we conclude the existence of a unit vector $\Theta_{\lambda, \omega}^+$ in \mathbb{R}^2 such that

$$\int_0^{\infty} \|U_{\lambda}(t, 0) \Theta_{\lambda, \omega}^+\|^{\delta} e^{\delta[\alpha_+(\lambda, \omega) - \varepsilon/2]t} dt < +\infty.$$

If we set $r(t) = \|U_{\lambda}(t, 0) \Theta_{\lambda, \omega}^+\|$, then the function $r(t)$ satisfies Eq. (II.16) and Lemma III.1 implies the existence of a constant $k = k(\lambda, \omega, \varepsilon)$ such that

$$\|U_{\lambda}(t, 0) \Theta_{\lambda, \omega}^+\| \leq k e^{-[\alpha_+(\lambda, \omega) - \varepsilon]t} \quad (\text{III.5})$$

for all $t \geq 0$. Now, for \mathbb{P} -almost all $\omega \in \Omega$, and σ^ω -almost all $\lambda \in R$, a well-known "folk theorem" on eigenfunction expansions of Schrödinger operators (see, for example, [3] for a proof), implies the existence of a real solution of $-y''(t) + (q(t, \omega) - \lambda)y(t) = 0$, say $y_{\lambda, \omega}(t)$, which is polynomially bounded both near $\pm\infty$. Hence, for ω in a set of full probability and for those λ 's in A we must have

$$\Theta_{\lambda, \omega}^- = \begin{bmatrix} y_{\lambda, \omega}(0) \\ y'_{\lambda, \omega}(0) \end{bmatrix} = \Theta_{\lambda, \omega}^+$$

at least up to a scalar multiplication, because of our assumption (i) and (III.5). Consequently, if we set $r_{\lambda, \omega}(t) = [y_{\lambda, \omega}(t)^2 + y'_{\lambda, \omega}(t)^2]^{1/2}$ for the amplitude of this solution, we have

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log r_{\lambda, \omega}(t) \leq -\alpha_+(\lambda, \omega) \quad (\text{III.6})$$

and

$$\lim_{t \rightarrow -\infty} \frac{1}{t} \log r_{\lambda, \omega}(t) = \alpha_-(\lambda, \omega). \quad (\text{III.7})$$

Relations (III.6) and (III.7) show in particular that the generalized eigenfunctions $y_{\lambda, \omega}(t)$ are actually L^2 -eigenfunctions, and since this is true for σ^ω -almost all $\lambda \in A$, this proves that the spectrum of $H(\omega)$ in A (if any) is pure point. They also show the exponential fall off claimed in the statement of the lemma. ■

We come now to the study of classes of stochastic processes (random potentials for us) $\{q(t, \omega); t \in \mathbb{R}, \omega \in \Omega\}$ giving rise to spectra of mixed types.

THEOREM II.4. *Let $\{X_t; t \geq 0\}$ be the stationary process of Brownian motion on a compact connected Riemannian manifold \mathcal{X} , let F be a real valued Morse function on \mathcal{X} such that $\inf F = -1$ and $\sup F = +1$ and let g be a continuous function on \mathbb{R} which satisfies $0 \leq g(t) \leq 1$ for $t < 0$, $g(t) = 1$ for $t \geq 0$, and $\int_{-\infty}^{\infty} g(t) dt < +\infty$. Then if we set*

$$\begin{aligned} q(t, \omega) &= g(t) q_1(t, \omega), & \text{for } t < 0, \quad \omega \in \Omega \\ &= g(t) F(X_t(\omega)), & \text{for } t \geq 0, \quad \omega \in \Omega, \end{aligned}$$

where $\{q_1(t, \omega); t < 0, \omega \in \Omega\}$ is any measurable stochastic process with bounded sample paths bounded below by -1 which is stochastically independent of $\{X_t; t \geq 0\}$, then there exists a set of full probability on which the following properties hold:

(a) *the spectrum of the unique self-adjoint extension in $L^2(\mathbb{R})$, say $H(\omega)$, of the operator defined for $f \in C_0^\infty(\mathbb{R})$ by $[H(\omega)f](t) = -f''(t) + q(t, \omega)f(t)$ is equal to $[-1, \infty)$.*

(b) *in $(0, \infty)$ this spectrum is purely absolutely continuous.*

(c) *in $(-1, 0)$ this spectrum is pure point and the eigenfunctions of the dense set of eigenvalues decay exponentially with rates of exponential fall off at $-\infty$ and $+\infty$ given by the upper Ljapunov indexes of the corresponding ordinary differential equations.*

Before proceeding to the proof we would like to emphasize several points relative to the statement of Theorem II.4. First, the constants used play essentially no role. Second, we will see in the proof below that we need only the conditional independence of $\{q_1(t); t < 0\}$ and $\{X_t; t \geq 0\}$ given X_0 . Finally, even though the stochastic process $\{q(t); t \in \mathbb{R}\}$ is not necessary stationary under the above assumptions, it is possible to show that 0 (and in fact any real number) is almost surely not an eigenvalue.

For the sake of completeness we recall the definition of a Morse function. A C^∞ -function F on a manifold \mathcal{X} is said to be a Morse function if there exists an integer n_0 such that for each $x \in \mathcal{X}$ we can find an integer $k \leq n_0$ for which $d^k F(x) \neq 0$.

Proof. If we fix ω in a set of probability 1 on which the assumptions of the theorem are satisfied, the function $q(t, \omega)$ is integrable near $-\infty$ and satisfies the assumptions of Corollary II.5. Hence the spectrum of $H(\omega)$ contains $(0, \infty)$ and is purely absolutely continuous there. This takes care of claim (b). Moreover the potential function $q(t, \omega)$ is bounded below by -1 so that the spectrum of $H(\omega)$ is contained in $[-1, \infty)$. Moreover, the ergodic properties of the process $\{X_t; t \geq 0\}$ show that \mathbb{P} -almost surely for each $\lambda \in [-1, 1]$, for each of its neighborhood N_λ and for each $L > 0$, there exists at least one interval I in $[0, \infty)$ of length greater than L and such that $F(X_t) \in N_\lambda$ for all $t \in I$, and by a classical argument this implies that the spectrum contains $[-1, +1]$ (see [2, 9]). This takes care of claim (a). To complete the proof we check the assumptions of Lemma II.2 for every open interval A whose closure is contained in $(-1, 0)$ and precise the information it gives on the exponential fall off of the eigenfunctions by Remark III.3. Checking conditions (i) is a simple exercise on the asymptotic behavior of the solutions of $-y''(t) + (q(t) - \lambda)y(t) = 0$ when $q(t)$ is integrable near $-\infty$ and $\lambda < 0$ (see, for example, [3, Chap. I]), condition (ii) is argued in Remark III.3 and condition (iii) is proved in [2, Lemma 3.3] with $\tau = 0$. ■

THEOREM III.5. *Let $\{X_t; t \geq 0\}$ be the stationary process of Brownian motion on a compact connected Riemannian manifold \mathcal{X} and let F be a real-valued Morse function on \mathcal{X} such that $\inf F = -\varepsilon$ and $\sup F = \varepsilon$ for some $\varepsilon > 0$. Let $q_1(t)$ be a periodic bounded and measurable real-valued function on \mathbb{R} and let us set for any $\omega \in \Omega$*

$$\begin{aligned} q(t, \omega) &= q_1(t), & \text{if } t < 0 \\ &= q_1(t) + F(X_t(\omega)), & \text{if } t \geq 0. \end{aligned}$$

Then there exists a set of full probability on which the following properties hold:

(a) *The spectrum of the unique self-adjoint extension in $L^2(\mathbb{R})$, say $H(\omega)$, of the operator defined for $f \in C_0^\infty(\mathbb{R})$ by $[H(\omega)f](t) = -f''(t) + q(t, \omega)f(t)$ is equal to $\bigcup_i [a_i - \varepsilon, b_i + \varepsilon]$ where $\bigcup_i [a_i, b_i]$ is the spectrum of the nonrandom self-adjoint operator $H_1 = -(d^2/dt^2) + q_1(t)$ in $L^2(\mathbb{R})$.*

(b) *This spectrum is purely absolutely continuous in $\bigcup_i [a_i, b_i]$.*

(c) *this spectrum is pure point in $\bigcup_i [a_i - \varepsilon, b_i + \varepsilon] \setminus \bigcup_i [a_i, b_i]$ and the corresponding eigenfunctions decay exponentially with rates of*

exponential fall off at $-\infty$ and $+\infty$ given by the upper Ljapunov indexes of the corresponding differential equations.

Proof. Let $\lambda = \lambda' + \varepsilon'$ with $\lambda' \in \bigcup_i [a_i, b_i]$ and $|\varepsilon'| < \varepsilon$. Since λ' is in the essential spectrum of H_1 , for each integer n we can find a function $\psi_n \in C_0^\infty(\mathbb{R})$ which is normalized in $L^2(\mathbb{R})$ and such that

$$\|-\psi_n'' + (q_1 - \lambda') \psi_n\| \leq n^{-1}. \quad (\text{III.8})$$

The ergodic properties of our stochastic process $\{X_t; t \geq 0\}$ are such that \mathbb{P} -almost surely, for each integer $n > 1$ there exists in $[0, \infty)$ an interval of length greater than the diameter of ψ_n plus twice the period of $q_1(t)$ on which we have

$$|F(X_t) - \varepsilon'| \leq n^{-1}. \quad (\text{III.9})$$

This means that we can always translate ψ_n (which will not affect (III.8)) in order to allow us to assume that ψ_n is supported in a region where (III.9) is satisfied. Now we have

$$\begin{aligned} \|-\psi_n'' + (q - \lambda) \psi_n\| &\leq \|-\psi_n'' + (q_1 - \lambda') \psi_n\| \\ &\quad + \|(F(X_t) - \varepsilon') \psi_n\| \leq 2n^{-1} \end{aligned}$$

and this takes care of (a). Statement (b) follows directly Corollary II.6 so that we are left with proving (c). In order to do that we apply Lemma III.2 to any bounded open interval A contained in $\bigcup_i [a_i - \varepsilon, b_i + \varepsilon] \setminus \bigcup_i [a_i, b_i]$. Property (i) follows well-known deterministic results since λ is in a so-called gap of the spectrum of H (see, for example, [3; Chap. II; 9]). Now we claim that the computations of [2 Sect. 3] and especially Lemma 3.3 can be performed by restricting the continuous variable $t \geq 0$ to take only values equal to integer multiples of the period of $q_1(t)$ and using Floquet's theory for periodic linear differential equations (see [5, 9]) to control the intermediate values of t . We refrain from reproducing the details of the proof here because the computations are lengthy, involved, and most of all do not contain any new idea. They take care of (ii) and (iii) of Lemma III.2. ■

ACKNOWLEDGMENTS

Part of this work was done while the author was visiting the Mathematical Institute of Aarhus University and the "Centre de Physique Théorique du C.N.R.S." in Marseille. The author would like to thank Palle T. Jorgensen (Aarhus University) for his kind invitation and both institutions for their warm hospitality. Finally we would like to thank Barry Simon for pointing out to us the application contained in Corollary II.8.

REFERENCES

1. F. BENTOSELA *et al.*, Schrödinger operators with electric field and random or deterministic potential, *Comm. Math. Phys.* **88** (1983), 387–397.
2. R. CARMONA, Exponential localization in one dimensional disordered systems, *Duke Math. J.* **49** (1982), 191–213.
3. R. CARMONA, Lecture notes, in preparation.
4. E. A. CODDINGTON AND N. LEVINSON, "Theory of Ordinary Differential Equations," McGraw-Hill, New York, 1955.
5. W. A. COPPEL, "Stability and Asymptotic Behavior of Differential Equations," Heath Math. Monographs, Boston, 1965.
6. W. CRAIG AND B. SIMON, Subharmonicity of the Lyapunov index. in *Duke Math. J.*, to appear.
7. E. B. DAVIES AND B. SIMON, Scattering theory for systems with different spacial asymptotics on the left and right, *Comm. Math. Phys.* (1978), 277–301.
8. E. I. DINABURG AND YA. G. SINAI, The one dimensional Schrödinger equation with a quasiperiodic potential, *Functional Anal. Appl.* **9** (1976), 279–289.
9. N. DUNFORD AND J. T. SCHWARTZ, "Linear Operators II, Spectral Theory," Wiley, New York, 1963.
10. R. S. P. EASTHAM, "The Spectral Theory of Periodic Differential Equations, Scottish Academic Press, London/New York, 1973.
11. I. JA. GOL'SHEID, S. A. MOLČANOV, AND L. A. PASTUR, A pure point spectrum of the stochastic one dimensional Schrödinger equation, *Functional Anal. Appl.* **11** (1977), 1–10.
12. H. KUNZ ET B. SOUILLARD, Sur le spectre des opérateurs aux différences finies aléatoires, *Comm. Math. Phys.* **78** (1980), 201–246.
13. B. M. LEVITAN AND I. S. SARGSJAN, "Introduction to Spectral Theory," Transl. of Math. Monographs, Vol. 39., Amer. Math Soc. Providence, R. I., 1975.
14. S. A. MOLČANOV, The structure of eigenfunctions of one dimensional unordered structures, *Math. USSR-Izv.* **12** (1978), 69–101.
15. G. ROYER, Etude des opérateurs de Schrödinger à potentiel aléatoire en dimension un, *Bull. Soc. Math. France* **110** (1982), 27–48.
16. I. SCH'NOL, On the behavior of Schrödinger equation, *Mat. Sb.* **42** (1957), 273–286. [Russian]
17. B. SIMON, Schrödinger semigroups, *Bull. Amer. Math. Soc.* **7** (1982), 447–526.